Exercise 6C: Solutions

1 If all three angles are less than 60°, then the sum of interior angles of the triangle would be less than 180°. This is a contradiction as the sum of interior angles is exactly 180°.

2 Suppose there is some least positive rational number $\frac{p}{q}$. Then since,

$$\frac{p}{2q} < \frac{p}{q},$$

there is some lesser positive rational number, which is a contradiction. Therefore, there is no least positive rational number.

3 Suppose that $\sqrt{p}$ is an integer. Then

$$\sqrt{p} = n,$$

for some $n \in \mathbb{Z}$. Squaring both sides gives

$$p = n^2.$$

Since $n \neq 1$, this means that p has three factors: $1, n$ and n^2. This is a contradiction since every prime number has exactly two factors.

4 Suppose that x is rational so that $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. Then,

$$\begin{aligned} & 3^x = 2 \\ \Rightarrow \quad & 3^{\frac{p}{q}} = 2 \\ \Rightarrow \quad & \left(3^{\frac{p}{q}}\right)^q = 2^q \\ \Rightarrow \quad & 3^p = 2^q \end{aligned}$$

The left hand side of this equation is odd, and the right hand side is even. This gives a contradiction, so x is not rational.

5 Suppose that $\log_2 5$ is rational so that $\log_2 5 = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. Then,

$$\begin{aligned} & 2^{\frac{p}{q}} = 5 \\ \Rightarrow \quad & 2^{\frac{p}{q}} = 5 \\ \Rightarrow \quad & \left(2^{\frac{p}{q}}\right)^q = 5^q \\ \Rightarrow \quad & 2^p = 5^q \end{aligned}$$

The left hand side of this equation is odd, and the right hand side is even. This gives a contradiction, so x is not rational.

6 Suppose the contrary, so that $\sqrt{x}$ is rational. Then

$$\sqrt{x} = \frac{p}{q},$$

where $p, q \in \mathbb{Z}$. Then, squaring both sides of the equation gives,

$$x = \frac{p^2}{q^2},$$

where $p^2, q^2 \in \mathbb{Z}$. Therefore, x is rational, which is a contradiction.

7 Suppose, on the contrary that $a + b$ is rational. Then

$$b = \overbrace{(a+b)}^{\text{rational}} - \overbrace{b.}^{\text{rational}}$$

Therefore, b is the difference of two rational numbers, which is rational. This is a contradiction.

8 Suppose b and c are both natural numbers. Then

$$c^2 - b^2 = 4$$
$$(c-b)(c+b) = 4.$$

The only factors of 4 are $1, 2$ and 4. And since $c+b > c-b$,

$$c - b = 1 \text{ and } c + b = 4.$$

Adding these two equations gives $2c = 5$ so that $c = \frac{2}{5}$, which is not a whole number.

9 Suppose that there are two different solutions, x_1 and x_2. Then,

$$ax_1 + b = c \text{ and } ax_2 + b = c.$$

Equating these two equations gives,

$$\begin{aligned} ax_1 + b &= ax_2 + b \\ ax_1 &= ax_2 \\ x_1 &= x_2, \qquad (\text{since } a \neq 0) \end{aligned}$$

which is a contradiction since the two solutions were assumed to be different.

10a Every prime $p > 2$ is odd since if it were even then p would be divisible by 2.

b Suppose there are two primes p and p such that $p + q = 1001$. Then since the sum of two odd numbers is even, one of the primes must be 2. Assume $p = 2$ so that $q = 999$. Since 999 is not prime, this gives a contradiction.

11a Suppose that

$$42a + 7b = 1.$$

Then

$$7(6a + b) = 1.$$

This implies that 1 is divisible by 7, which is a contradiction since the only factor of 1 is 1.

b Suppose that

$$15a + 21b = 2.$$

Then

$$3(5a + 7b) = 2.$$

This implies that 2 is divisible by 3, which is a contradiction since the only factors of 2 are 1 and 2.

12a Contrapositive: If n is not divisible by 3, then n^2 is not divisible by 3.
Proof: If n is not divisible by 3 then either $n = 3k + 1$ or $n = 3k + 2$.
(Case 1) If $n = 3k + 1$ then,

$$\begin{aligned} n^2 &= (3k+1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \end{aligned}$$

is not divisible by 3.
(Case 2) If $n = 3k + 2$ then,

$$\begin{aligned} n^2 &= (3k+2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 9k^2 + 12k + 3 + 1 \\ &= 3(3k^2 + 4k + 1) + 1 \end{aligned}$$

is not divisible by 3.

b This will be a proof by contradiction. Suppose $\sqrt{3}$ is rational so that $\sqrt{3} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. We can assume that p and q have no common factors (or else they could be cancelled). Then,

$$\begin{aligned} & p^2 = 3q^2 \quad (1) \\ \Rightarrow \quad & p^2 \text{ is divisible by } 3 \\ \Rightarrow \quad & p \text{ is divisible by } 3 \\ \Rightarrow \quad & p = 3k \text{ for some } k \in \mathbb{N} \\ \Rightarrow \quad & (3k)^2 = 3q^2 (\text{substitituting into } (1)) \\ \Rightarrow \quad & 3q^2 = 9k^2 \\ \Rightarrow \quad & q^2 = 3k^2 \\ \Rightarrow \quad & q^2 \text{ is divisible by } 3 \\ \Rightarrow \quad & q \text{ is divisible by } 3. \end{aligned}$$

So p and q are both divisible by 3, which contradicts the fact that they have no factors in common.

13a Contrapositive: If n is odd, then n^3 is odd.
Proof: If n is odd then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^3 &= (2k+1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1 \end{aligned}$$

is odd. Otherwise, we can simply quote the fact that the product of 3 odd numbers will be odd.

b This will be a proof by contradiction. Suppose $\sqrt[3]{2}$ is rational so that $\sqrt[3]{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. We can assume that p and q have no common factors (or else they could be cancelled). Then,

$$\begin{aligned} & p^3 = 2q^3 \quad (1) \\ \Rightarrow \quad & p^3 \text{ is divisible by } 2 \\ \Rightarrow \quad & p \text{ is divisible by } 2 \\ \Rightarrow \quad & p = 2k \text{ for some } k \in \mathbb{N} \\ \Rightarrow \quad & (2k)^3 = 2q^3 (\text{substitituting into } (1)) \\ \Rightarrow \quad & 2q^3 = 8k^3 \\ \Rightarrow \quad & q^3 = 4k^3 \\ \Rightarrow \quad & q^3 \text{ is divisible by } 2 \\ \Rightarrow \quad & q \text{ is divisible by } 2. \end{aligned}$$

So p and q are both divisible by 2, which contradicts the fact that they have no factors in common.

14 This will be a proof by contradiction, so we suppose there is some $a, b \in \mathbb{Z}$ such that

$$\begin{aligned} & a^2 - 4b - 2 = 0 \\ & \quad \Rightarrow a^2 = 4b + 2 \\ & \quad \Rightarrow a^2 = 2(2b+1) \quad (1) \end{aligned}$$

which means that a^2 is even. However, this implies that a is even, so that $a = 2k$, for some $k \in \mathbb{Z}$. Substituting this into equation (1) gives,

$$\begin{aligned} (2k)^2 &= 2(2b+1) \\ 4k^2 &= 2(2b+1) \\ 2k^2 &= 2b+1 \\ 2k^2 - 2b &= 1 \\ 2(k^2 - b) &= 1. \end{aligned}$$

This implies that 1 is divisible by 2, which is a contradiction since the only factor of 1 is 1.

15a Suppose on the contrary, that $a > \sqrt{n}$ and $b > \sqrt{n}$. Then

$$ab > \sqrt{n}\sqrt{n} = n,$$

which is a contradiction since $ab = n$.

b If 97 were not prime then we could write $97 = ab$ where $1 < a < b < n$. By the previous question, we know that

$$a \leq \sqrt{97} < \sqrt{100} = 10.$$

Therefore a is one of

$$\{2, 3, 4, 5, 6, 7, 8, 9\}.$$

However 97 is not divisible by any of these numbers, which is a contradiction. Therefore, 97 is a prime number.

16a Let $m = 4n + r$ where $r = 0, 1, 2, 3$.
($r = 0$) We have,

$$\begin{aligned} m^2 &= (4n)^2 \\ &= 16n^2 \\ &= 4(4n^2) \end{aligned}$$

is divisible by 4.
($r = 1$) We have,

$$\begin{aligned} m^2 &= (4n+1)^2 \\ &= 16n^2 + 8n + 1 \\ &= 4(4n^2 + 2n) + 1 \end{aligned}$$

has a remainder of 1.
($r = 2$) We have,

$$\begin{aligned} m^2 &= (4n+2)^2 \\ &= 16n^2 + 16n + 4 \\ &= 4(4n^2 + 4n + 1) \end{aligned}$$

is divisible by 4.
($r = 3$) We have,

$$\begin{aligned} m^2 &= (4n+3)^2 \\ &= 16n^2 + 24n + 9 \\ &= 16n^2 + 24n + 8 + 1 \\ &= 4(4n^2 + 6n + 2) + 1 \end{aligned}$$

has a remainder of 1.

Therefore, the square of every integer is divisible by 4 or leaves a remainder of 1.

b Suppose the contrary, so that both a and b are odd. Then $a = 2k + 1$ and $b = 2m + 1$ for some $k, m \in \mathbb{Z}$. Therefore,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (2k+1)^2 + (2m+1)^2 \\ &= 4k^2 + 4k + 1 + 4m^2 = 4m + 1 \\ &= 4(k^2 + m^2 + k + m) + 2. \end{aligned}$$

This means that c^2 leaves a remainder of 2 when divided by 4, which is a contradiction.

17a Suppose by way of contradiction either $a \neq c$ or $b \neq d$. Then clearly both $a \neq c$ and $b \neq d$. Therefore,

$$\begin{aligned} a + b\sqrt{2} &= c + d\sqrt{2} \\ (b-d)\sqrt{2} &= c - a \\ \sqrt{2} &= \frac{c-a}{b-d} \end{aligned}$$

Since $\frac{c-a}{b-d} \in \mathbb{Q}$, this contradicts the irrationality of $\sqrt{2}$.

b Squaring both sides gives,

$$\begin{aligned} 3 + 2\sqrt{2} &= (c + d\sqrt{2})^2 \\ 3 + 2\sqrt{2} &= c^2 + 2cd\sqrt{2} + 2d^2 \\ 3 + 2\sqrt{2} &= c^2 + 2d^2 + 2cd\sqrt{2} \end{aligned}$$

Therefore

$$\begin{aligned} c^2 + 2d^2 &= 3 \qquad (1) \\ cd &= 1 \qquad (2) \end{aligned}$$

Since c and d are integers, this implies that $c = d = 1$.

18 There are many ways to prove this result. We will take the most elementary approach (but not the most elegant). Suppose that

$$ax^2 + bx + c = 0 \qquad (1)$$

has a rational solution, $x = \frac{p}{q}$. We can assume that p and q have no factors in common (or else we could cancel). Equation (1) then becomes

$$\begin{aligned} ax^2 + bx + c &= 0 \\ a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c &= 0 \\ ap^2 + bpq + cq^2 &= 0 \qquad (2) \end{aligned}$$

Since p and q cannot both be even, we need only consider three cases.

(Case 1) If p is odd and q is odd then equation (2) is of the form

$$\text{odd} + \text{odd} + \text{odd} = \text{odd} = 0.$$

This is not possible since 0 is even.

(Case 2) If p is odd and q is even then equation (2) is of the form

$$\text{odd} + \text{even} + \text{even} = \text{odd} = 0. [t]$$

This is not possible since 0 is even.

(Case 3) If p is even and q is odd then equation (2) is of the form

$$\text{even} + \text{even} + \text{odd} = \text{odd} = 0.$$

This is not possible since 0 is even.